Calculus of variations

Euler–Lagrange equation

- q S $q\left(t\right)$
. S

\begin{equation} \displaystyle S(q) = \int_a^b L(t,q(t),q'(t))\, \mathrm{d}t \end{equation}

. :

- q , :

\begin{align} q \colon [a, b] \subset \mathbb{R} & \to X \\ t & \mapsto x = q(t) \end{align}

q , $q\left(a\right) = x_a, q\left(b\right) = x_b$.

- q' q .

-

1 - .
.

f , $f\left(a\right) = c, f\left(b\right) = d$,
J .

\begin{equation} J = \int_a^b F(x,f(x),f'(x))\, dx. \,\! \end{equation}

F . (,
.)

f , , f , J
(f J) J .(f J)

f $g_\epsilon\left(x\right) = f\left(x\right)+\epsilon\eta\left(x\right)$. $\eta\left(x\right)$ $\eta\left(a\right) = \eta\left(b\right) = 0$. , f g J
.

\begin{equation} J(\epsilon) = \int_a^b F(x,g_\epsilon(x), g_\varepsilon'(x))\, dx. \,\! \end{equation}

J ϵ ,

\begin{equation} \frac{\mathrm{d} J}{\mathrm{d} \varepsilon} = \int_a^b \frac{\mathrm{d}F}{\mathrm{d}\epsilon}(x,g_\varepsilon(x), g_\varepsilon'(x))\, dx. \end{equation}

,

\begin{equation} \frac{\mathrm{d}F}{\mathrm{d}\epsilon} = \frac{\partial F}{\partial

x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial F}{\partial g_\varepsilon}\frac{\partial g_\varepsilon}{\partial \varepsilon} + \frac{\partial F}{\partial g'_\varepsilon}\frac{\partial g'_\varepsilon}{\partial \varepsilon} = \eta(x) \frac{\partial F}{\partial g_\varepsilon} + \eta'(x) \frac{\partial F}{\partial g_\varepsilon'}. \end{equation}

\begin{equation} \frac{\mathrm{d} J}{\mathrm{d} \epsilon} = \int_a^b \left[\eta(x) \frac{\partial F}{\partial g_\varepsilon} + \eta'(x) \frac{\partial F}{\partial g_\varepsilon'} \, \right]\,dx. \end{equation}

$\epsilon = 0$ $g_\epsilon = f$, f J , $J'\left(0\right) = 0$,   .   ,

\begin{equation} J'(0) = \int_a^b \left[\eta(x) \frac{\partial F}{\partial f} + \eta'(x) \frac{\partial F}{\partial f'} \,\right]\,dx = 0. \end{equation}

         ,   .   .

\begin{equation} 0 = \int_a^b \left[\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} \right] \eta(x)\,dx + \left[\eta(x) \frac{\partial F}{\partial f'} \right]_a^b. \end{equation}

η ,

\begin{equation} 0 = \int_a^b \left[\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} \right] \eta(x)\,dx. \,\! \end{equation}

          ,   –   .

\begin{equation} 0 = \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'}. \end{equation}

2 $\left(a, y_a\right)$ $\left(b, y_b\right)$. L .

\begin{equation} L\left[f\right] = \int_a^b \sqrt{1 + f'\left(x\right)^2}\, dx \end{equation}

f $f\left(a\right) = y_a, f\left(b\right)=y_b$.

             –   , f

\begin{equation} 0 = -\frac{d}{dx}\frac{\partial }{\partial f'}\sqrt{1 + f'\left(x\right)^2} \end{equation}

            .   ,

\begin{equation} \begin{matrix} 0 &=& \frac{d}{dx}\frac{\partial }{\partial f'}\sqrt{1 + f'\left(x\right)^2} \\ &=& \frac{d}{dx}\frac{f'\left(x\right)}{\sqrt{1 + f'\left(x\right)^2}} \end{matrix} \end{equation}

O ,

\begin{equation} \frac{f'\left(x\right)}{\sqrt{1 + f'\left(x\right)^2}} = k \end{equation}

, $f'\left(x\right)$

.

\begin{equation} f'\left(x\right) = C \end{equation}

$f\left(x\right)=Cx + D$.